Cover Semantics for Predicate Extensions of Intuitionistic Modal Logics

Danya Rogozin
Institute for Information Transmission Problems, RAS, Moscow, Russia
29. November 2020

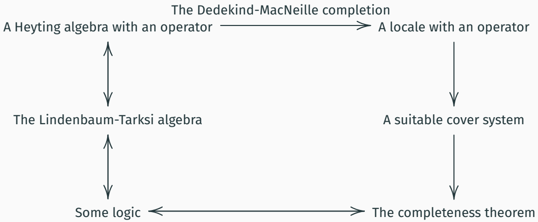
Foreword

Remarks

- We are going to develop several ideas by Goldblatt on semantics of intuitionistic modal predicate logics.
- This approach is based on the Dedekind-MacNeille completions and cover systems. Such an approach is striclty connected with Kripke-Joyal semantics.
- We consider modalities from the logic $\mathbf{IEL}^{-}$ algebraically as prenuclear operators from point-free topology and the theory of locales.

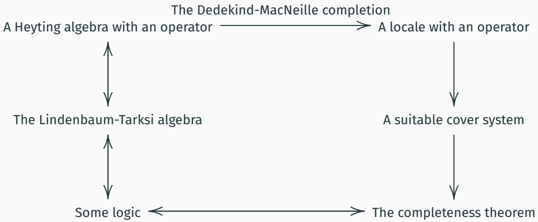
Completeness with respect to cover semantics

We have the following scheme:



Completeness with respect to cover semantics

We have the following scheme:



1. *The diagram above "commutes".*
2. All main technical tricks are related to the Dedekind-MacNeille completion.

The definition of a (complete) Heyting algebra

1. A *Heyting algebra* is a bounded distributive lattice $\mathcal{H} = \langle H, \wedge, \vee, \bot, \top \rangle$ with the binary operation $\Rightarrow$ such that the following equivalence holds:

$$a \wedge b \leq c \text{ iff } a \leq b \Rightarrow c$$

2. A complete Heyting algebra (locale) is a complete lattice $\mathcal{L} = \langle L, \wedge, \bigvee \rangle$ such that finite infima distribute over arbitrary suprema:

$$a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\} \text{ for each } B \subseteq L.$$

The definition of a (complete) Heyting algebra

1. A *Heyting algebra* is a bounded distributive lattice $\mathcal{H} = \langle H, \wedge, \vee, \bot, \top \rangle$ with the binary operation $\Rightarrow$ such that the following equivalence holds:

$$a \wedge b \leq c \text{ iff } a \leq b \Rightarrow c$$

2. A complete Heyting algebra (locale) is a complete lattice $\mathcal{L} = \langle L, \wedge, \bigvee \rangle$ such that finite infima distribute over arbitrary suprema:

$$a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\} \text{ for each } B \subseteq L.$$

Locales are about

1. A lattice-theoretic approximation of topogical spaces,
2. The complex algebra of an arbitrary intuitonistic Kripke frame is actually a locale.

A nucleus operator

Definition

A *nucleus* operator on a Heyting algebra $\mathcal{H}$ is a closure operator j distributing over finite infima

A nucleus operator

Definition

A *nucleus* operator on a Heyting algebra $\mathcal{H}$ is a closure operator j distributing over finite infima

Nuclei are about

- Fixpoints form a Heyting subalgebra.
- A nucleus is a (version of) geometric modality describing the notion of local truth. This aspect is based on the notion of a Lawvere-Tierney-Grothendieck topology.
- Dragalin-style semantics of intuitionistic and intermediate logics.
- As a modality, a nucleus is an algebraic analogue of the lax modality.
- We've already discussed the connection with the lax logic.

Before the strict definition, we discuss some background stuff:

Background stuff

- Cover semantics is based on the notion of local truth in topological/topos-theoretic terms.
- A statement φ is true at X (where X is a space or an open set), if X has an open cover for each member of which φ is true.
- The example is a property of a function being locally constant on some neighbourhood.
- Cover semantics is about abstract local truth with arbitrary cover systems (non-necessary topologically based).
- A statement φ is locally true at x (of some partial order) if x has a cover C such that φ is true at every point of C.

Let $\langle P, \leq \rangle$ be a poset and $\triangleright$ a a binary relation between P and $\mathcal{P}(P)$. $x \in P$ and $C \subseteq P$, then we say that x is *covered* by C (C is an x-cover), if $x \triangleright C$.

Let $\langle P, \leq \rangle$ be a poset and $\triangleright$ a a binary relation between P and $\mathcal{P}(P)$. $x \in P$ and $C \subseteq P$, then we say that x is *covered* by C (C is an x-cover), if $x \triangleright C$.

(strictly localic) cover system

A (strictly localic) *cover system* is a triple $\mathcal{S} = \langle P, \leq, \triangleright \rangle$ such that

1. (Existence) There exists an x-cover $C \subseteq \uparrow x$
2. (Transitivity) Let $x \triangleright C$ and for each $y \in C$ $y \triangleright C_y$, then $x \triangleright \bigcup_{y \in C} C_y$
3. (Refinement) If $x \leq y$, then $C \triangleright x$ implies that there exists an y-cover C' such that $C' \subseteq \uparrow C$
4. (the strict localic axiom) Every x-cover is included in $\uparrow x$.

Cover systems, due to [Goldblatt 2011]

Let $\mathcal{S} = \langle P, \leq, \triangleright \rangle$ be a strictly localic cover system.

j-operator

Define an operator $j : \mathcal{P}(P) \to \mathcal{P}(P)$ such that:

$$jX = \{x \in P \mid \exists C\, x \triangleright C \subseteq X\}$$

x is a *local member* X, if $x \in jX$. A subset X is *localised*, if $jX = X$. We call such a set a *proposition*.

Theorem [Goldblatt 2011]

1. If X is an up-set, so is jX. Moveover, j is a nucleus operator on $\mathbf{Up}(P, \leq)$.
2. The set of localised up-sets is a locale with $\bigvee_{i \in I} X_i = j(\bigcup_{i \in I} X_i)$.
3. Every locale is isomorphic to the locale of propositions of some cover system.

Dedekind-MacNeille completion

Given a bounded lattice $\mathcal{L}$, a *completion* of $\mathcal{L}$ is a complete lattice $\overline{\mathcal{L}}$ that contains $\mathcal{L}$ as a sublattice. A completion $\overline{\mathcal{L}}$ is called *Dedekind-MacNeille* if every element of $a \in \overline{\mathcal{L}}$ is both a join and meet of elements of $\mathcal{L}$:

$$a = \bigvee\{b \in \mathcal{L} \mid a \leq b\} = \bigwedge\{b \in \mathcal{L} \mid b \leq a\}.$$

If $\mathcal{L}$ is a Heyting algebra, then $\overline{\mathcal{L}}$ is a locale.

The Dedekind-MacNeille completion for Heyting algebras

Dedekind-MacNeille completion

Given a bounded lattice $\mathcal{L}$, a *completion* of $\mathcal{L}$ is a complete lattice $\overline{\mathcal{L}}$ that contains $\mathcal{L}$ as a sublattice. A completion $\overline{\mathcal{L}}$ is called *Dedekind-MacNeille* if every element of $a \in \overline{\mathcal{L}}$ is both a join and meet of elements of $\mathcal{L}$:

$$a = \bigvee \{b \in \mathcal{L} \mid a \leq b\} = \bigwedge \{b \in \mathcal{L} \mid b \leq a\}.$$

If $\mathcal{L}$ is a Heyting algebra, then $\overline{\mathcal{L}}$ is a locale.

Theorem

Every Heyting algebra is representable as a subalgebra of propositions of some strictly localic cover system.

Cover models

Let $\mathcal{S} = \langle P, \leq, \triangleright \rangle$ be a strictly localic cover system and let D be a non-empty set, a domain of individuals. Let V be a valuation function that maps each k-ary predicate letter P to $V(P) : D^k \to \mathrm{Prop}(\mathcal{S})$.

A D-assignment is an infinite sequence $\sigma = \langle \sigma_0, \sigma_1, \ldots, \sigma_n, \ldots \rangle$, where $\sigma_i \in D$ for each $i < \omega$. Such a D-assignment maps each variable x_i to the corresponding σ_i.

A structure $\mathfrak{M} = \langle \mathcal{S}, D, V \rangle$ is a *cover model*.

Cover models

Let $\mathcal{S} = \langle P, \leq, \triangleright \rangle$ be a strictly localic cover system and let D be a non-empty set, a domain of individuals. Let V be a valuation function that maps each k-ary predicate letter P to $V(P) : D^k \to \mathrm{Prop}(\mathcal{S})$.

A D-assignment is an infinite sequence $\sigma = \langle \sigma_0, \sigma_1, \ldots, \sigma_n, \ldots \rangle$, where $\sigma_i \in D$ for each $i < \omega$. Such a D-assignment maps each variable x_i to the corresponding σ_i.

A structure $\mathfrak{M} = \langle \mathcal{S}, D, V \rangle$ is a *cover model*.

The forcing relation

1. $\mathfrak{M}, x, \sigma \Vdash P(x_{n_1}, \ldots, x_{n_k})$ iff $x \in V(P)(\sigma_{n_1}, \ldots, \sigma_{n_k})$.
2. $\mathfrak{M}, x, \sigma \Vdash \bot$ iff $x \triangleright \emptyset$
3. $\mathfrak{M}, x, \sigma \Vdash \varphi \vee \psi$ iff there exists an x-cover C such that for each $y \in C$ $\mathfrak{M}, y, \sigma \Vdash \varphi$ or $\mathfrak{M}, y, \sigma \Vdash \psi$.
4. $\mathfrak{M}, x, \sigma \Vdash \exists x_n \varphi$ iff there exist an x-cover C and $d \in D$ such that for each $y \in C$ one has $\mathfrak{M}, y, \sigma(d/n) \Vdash \varphi$.

Completeness theorem for intuitionistic predicate logic

Theorem

Intuitionistic predicate logic is complete w.r.t. strictly localic models.

Completeness theorem for intuitionistic predicate logic

Theorem

Intuitionistic predicate logic is complete w.r.t. strictly localic models.

Based on the representation of the Dedekind-MacNeille completion of the Lindenbaum-Tarski algebra with some strictly localic cover system. We put variables as the set of domain individuals.

Completeness theorem for intuitionistic predicate logic

Theorem

Intuitionistic predicate logic is complete w.r.t. strictly localic models.

Based on the representation of the Dedekind-MacNeille completion of the Lindenbaum-Tarski algebra with some strictly localic cover system. We put variables as the set of domain individuals.

Alternatively, one may construct a strictly localic cover system on certian theories (ever weaker than prime ones). Such a construction (simpler than the Henkin construction) is also due to Goldblatt.

Let $\mathcal{S} = \langle P, \leq, \triangleright \rangle$ be a strictly localic cover system and $R \subseteq P \times P$ a binary relation on P.

Modal cover system

A structure $\mathcal{S} = \langle P, R, \leq, \triangleright \rangle$ is called modal cover system, if the following hold:

1. (Confluence) If $x \leq y$ and xRz, then there exists w such that yRw and $z \leq w$.
2. (Modal localisation) If there exists C such that $x \triangleright C \subseteq \langle R \rangle A$, then there exists $y \in R(x)$ with a y-cover included in X.

where $\langle R \rangle A = \{x \in S \mid \exists y \in A \, xRy\} = R^{-1}(A)$

Theorem

Let $\mathcal{L}$ be a locale and $f : \mathcal{L} \to \mathcal{L}$ a monotone operator, then $\langle \mathcal{L}, f \rangle$ is isomorphic to some algebra of propositions of some modal cover system.

Prenuclear operators, the origins

Theorem

Let $\mathcal{H}$ be a Heyting algebra, a *prenucleus* on $\mathcal{H}$ is an operator monotone $j : \mathcal{H} \to \mathcal{H}$ such that for each $a, b \in \mathcal{H}$:

1. $a \leq ja$
2. $ja \wedge b \leq j(a \wedge b)$.

A prenucleus is called multiplicative if it distributives over finite infima (an **IEL**$^-$-algebra). A multiplicative prenuclear algebra is called an **IEL**-algebra, if $j\bot = \bot$

Prenuclear operators, the origins

Theorem

Let $\mathcal{H}$ be a Heyting algebra, a *prenucleus* on $\mathcal{H}$ is an operator monotone $j : \mathcal{H} \to \mathcal{H}$ such that for each $a, b \in \mathcal{H}$:

1. $a \leq ja$
2. $ja \wedge b \leq j(a \wedge b)$.

A prenucleus is called multiplicative if it distributives over finite infima (an $\mathbf{IEL}^-$-algebra). A multiplicative prenuclear algebra is called an **IEL**-algebra, if $j\bot = \bot$

Prenuclear algebras are about

- Generating nucleus on a locale with transfinite sequence of prenuclei.
- A nuclear reflection, an approximation of a nucleus on a locale with prenuclei, to define the join of nuclei.
- A multiplicative prenuclear operator also reminds of the $\mathbf{IEL}^-$ modality.

The Dedekind-MacNeille completion for prenuclear algebras

- Here we note that all those classes are closed under the Dedekind-MacNeille completion.
- Let us drop the details, it is not so complicated, but tricky anyway.

Prenuclear cover systems

Let $\mathcal{S} = \langle S, \preceq, \triangleright, R \rangle$ be a modal cover system, then $\mathcal{S}$ is called prenuclear, if the following two conditions hold:

1. R is reflexive.
2. Let $x, y \in S$ such that xRy, then there exists $z \in \uparrow y$ such that $x \leq z$ and $x \in R(z)$.

Prenuclear cover systems and their variations

Prenuclear cover systems

Let $\mathcal{S} = \langle S, \preceq, \triangleright, R \rangle$ be a modal cover system, then $\mathcal{S}$ is called prenuclear, if the following two conditions hold:

1. R is reflexive.
2. Let $x, y \in S$ such that xRy, then there exists $z \in \uparrow y$ such that $x \leq z$ and $x \in R(z)$.

A prenuclear cover system is called multiplicative (an $\mathbf{IEL}^-$-cover system), if the following hold:

1. R is serial, that is, for each $x \in S$ there exists $y \in S$ such that xRy.
2. if xRy and xRz then there exists $w \in \uparrow x \cap \uparrow y$ such that xRw.
3. Let $x, y \in S$ such that xRy, then there exists $z \in \uparrow y$ such that $x \leq z$ and $x \in R(z)$.

Prenuclear cover systems

Let $\mathcal{S} = \langle S, \preceq, \triangleright, R\rangle$ be a modal cover system, then $\mathcal{S}$ is called prenuclear, if the following two conditions hold:

1. R is reflexive.
2. Let $x, y \in S$ such that xRy, then there exists $z \in \uparrow y$ such that $x \leq z$ and $x \in R(z)$.

A prenuclear cover system is called multiplicative (an **IEL**$^-$-cover system), if the following hold:

1. R is serial, that is, for each $x \in S$ there exists $y \in S$ such that xRy.
2. if xRy and xRz then there exists $w \in \uparrow x \cap \uparrow y$ such that xRw.
3. Let $x, y \in S$ such that xRy, then there exists $z \in \uparrow y$ such that $x \leq z$ and $x \in R(z)$.

A multiplicative prenuclear cover system is called an **IEL**-cover system, if for each $x, y \in S$ if xRy and $y \triangleright \emptyset$ implies $x \triangleright \emptyset$.

The representation theorem for prenuclear algebras

Here we apply our Dedekind-MacNeille statement for the second item of the following theorem:

Theorem, [D. R. 2020]

1. Every complete prenuclear algebra (as well as a multiplicative or an **IEL** one) is isomorphic to some algebra of localised up-set of the corresponding cover system.
2. Every prenuclear algebra (as well as a multiplicative or **IEL** one) is isomorphic to some subalgebra of localised up-set of the corresponding cover system.

The underlying logic for us is the following

$\mathbf{IEL}^{-}_{-}$ and its extensions

- **IPC** axioms
- $\varphi \to \bigcirc\varphi$
- $\varphi \wedge \bigcirc\psi \to \bigcirc(\varphi \wedge \psi)$
- The rules are the Modus Ponens and $\bigcirc$-monotonicity: from $\varphi \to$

So,

- $\mathbf{IEL}^{-} = \mathbf{IEL}^{-}_{-} \oplus \bigcirc(\varphi \wedge \psi) \leftrightarrow (\bigcirc\varphi \wedge \bigcirc\psi) \oplus \bigcirc\top \leftrightarrow \top$,
- $\mathbf{IEL} = \mathbf{IEL}^{-} \oplus \neg\bigcirc\bot$.

$\mathbf{IEL}^{-}_{-}$ and its extensions

Let $\mathbf{L} \in \{\mathbf{IEL}^{-}_{-}, \mathbf{IEL}^{-}, \mathbf{IEL}\}$, then **QL** is a predicate extension of **L**. The signature is purely relational with no constants and function letters.

The completeness theorem

Let $\mathbb{C}$ be a class of modal cover systems, then $\mathrm{Log}(\mathbb{C})$ is its logic.

The completeness theorem

Let $\mathbb{C}$ be a class of modal cover systems, then $\mathrm{Log}(\mathbb{C})$ is its logic.

Soundness, [Goldblatt 2011]

Let $\mathbb{C}$ be a class of modal cover systems, then $\mathrm{Log}(\mathbb{C})$ contains the first-order intuitionistic neighbourhood modal logic (intuitionistic predicate logic plus the $\bigcirc$-monotonicity rule)

The completeness theorem

Let $\mathbb{C}$ be a class of modal cover systems, then $\mathrm{Log}(\mathbb{C})$ is its logic.

Soundness, [Goldblatt 2011]

Let $\mathbb{C}$ be a class of modal cover systems, then $\mathrm{Log}(\mathbb{C})$ contains the first-order intuitionistic neighbourhood modal logic (intuitionistic predicate logic plus the $\bigcirc$-monotonicity rule)

The completeness theorem, [D. R. 2020]

- Let $\mathbb{C}$ be the class of all prenuclear cover systems, then $\mathrm{Log}(\mathbb{C}) = \mathbf{QIEL}^{-}_{-}$.
- Let $\mathbb{C}$ be the class of all multiplicative prenuclear cover systems, then $\mathrm{Log}(\mathbb{C}) = \mathbf{QIEL}^{-}$.
- Let $\mathbb{C}$ be the class of all **IEL** cover systems, then $\mathrm{Log}(\mathbb{C}) = \mathbf{QIEL}$.

Thank you for your kind attention!